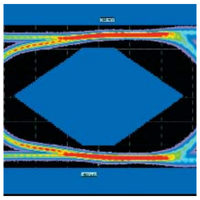

T E C H N I C A L   B R I E F

# The Relationship Between Variance of Time Measurements and the Autocorrelation Function

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*(with some additional comments and edits by*
*Dr. Martin Miller, LeCroy Corporation, Abstract by Dr. Miller*
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*Both Dr. Chow and Dr. Miller were motivated to understand the viability and justification of the methods prescribed by WaveCrest™ to obtain periodic Jitter measurement results from pure time-time measurements. While it was apparent that the spectrum of the sdev vs. cycle contained components at contributors to periodic jitter, the basis for this fact was not mathematically substantiated in any publication easily obtained. The purveyors of this method were not forthcoming to anyone (customer or competitor). This simple relationship was easily uncovered by Dr. Chow in his personal search for justification of the method. Simultaneously, Dr. Miller was trying to "correlate" results from LeCroy Corporation jitter instrumentation to WaveCrest™ instrumentation, and contacted Dr. Chow privately.*

*In the interest of spreading the "word" and understanding of this technique, they decided this derivation is worth sharing with the general public.*

## Introduction

In spectral analysis, the standard "textbook" technique for finding frequency-domain spectra from time-domain measurements is the Blackman-Tukey method[1]. This method finds the power spectral density (PSD) by performing a Fourier transform on the autocorrelation function of the observed variable. The discrete autocorrelation function is given by

$$R_x(l)^2 = \frac{1}{N}\sum_{n=1}^{N-1}\left[x_n x_{n+l}\right] \tag{1}$$

This is the method used by Wavecrest™ for finding the jitter modulation spectra of clock and data signals. They refer to this measurement technique as Accumulated Time Analysis™. For more information on Accumulated Time Analysis™, please refer to their Web site, www.wavecrest.com.

In this document, we show that the measured variance of accumulated jitter is equivalent[2] to the autocorrelation function of jitter.

## Definitions

For simplicity in this proof, we use a clock. Application of this methodology to data signals is discussed in the section on Jitter Spectra. Consider a clock signal with transition edges at certain times, as shown in Figure 1.

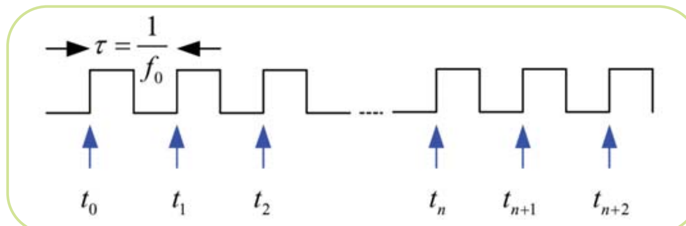


***Figure 1.*** *Timing of clock edges and notation*

Let $t_n$ be the time at which the $n$th clock edge occurs. The time $t_n$ is the time at which the transition happens. The clock speed is $f_0$ and the ideal clock period is $\tau$. The measurements for each edge are denoted by an integer $i$ for a total of $M$ measurements (e.g., $t_{n,i}$). The mean values of $t_n$ and $t_n^2$ are given by

$$\overline{t_n} = \frac{1}{M}\sum_{i=1}^{M} t_{n,i} \text{ and } \overline{t_n^2} = \frac{1}{M}\sum_{i=1}^{M} t_{n,i}^2 \tag{2}$$

Since the labeling[3] $n$ is arbitrary, the expectation value of any edge $t_n$, where $\langle t \rangle$ is independent of $n$, is given by

$$\langle t \rangle = \overline{t_n^2} - \left(\overline{t_n}\right)^2 \tag{3}$$

For edge-to-edge measurements across $n$ periods, the mean value is equal to $n$ times the ideal clock period, or

$$n\tau = \overline{t_n} - \overline{t_0} \tag{4}$$

This should be quite evident for a clock signal. For the case of data patterns, Equation 4 assumes that effects from duty cycle distortions (DCD) and inter-symbol interference (ISI) are negligible.

Let the mean position of the starting edge be the origin (all $t_0 = 0$), and so

[1] Blackman, R. B., and Tukey, J. W., *The Measurement of Power Spectra*, Dover, 1958.

[2] To within a constant

[3] If the terminology "labeling" seems too abstract, then consider that the measurement of any such values may be assumed to be incoherent with any systematic behavior. A case that violates this assumption would include sampling the system synchronously with *any* periodic interference.

$$\overline{t_0} = 0 \tag{5}$$

Substitution into Equation 4 gives us,

$$t_n = n\tau \tag{6}$$

which should not be so surprising. The expectation values become

$$\langle t \rangle = \overline{t_n^2} - (n\tau)^2 \text{ and } \langle t \rangle = \overline{t_0^2} \tag{7}$$

## VARIANCE

An edge-to-edge measurement across $n$ periods (recalling Equation 5) is given by

$$\Delta t_n = t_n - t_0 = t_n \tag{8}$$

The variance of repeated edge-to-edge measurements is given by

$$\begin{aligned} \sigma_n^2 &= \frac{1}{M-1}\sum_{i=1}^{M}\left[\Delta t_{n,i} - \overline{\Delta t_n}\right]^2 \\ &= \frac{1}{M-1}\sum_{i=1}^{M}\left[t_{n,i} - t_{0,i} - \overline{t_n} + \overline{t_0}\right]^2 \\ &= \frac{1}{M-1}\sum_{i=1}^{M}\left[t_{n,i} - t_{0,i} - n\tau\right]^2 \end{aligned} \tag{9}$$

where $n$ is the number of periods spanned by the measurement. Expanding the square inside the summation, we get

$$\sigma_n^2 = \frac{1}{M-1}\sum_{i=1}^{M}\left[t_{n,i}{}^2 + t_{0,i}{}^2 + (n\tau)^2 - 2t_{n,i}n\tau + 2t_{0,i}n\tau - 2t_{n,i}t_{0,i}\right] \tag{10}$$

Performing the summation and using the definition of mean values from Equation 2, we get

$$\sigma_n{}^2 = \frac{M}{M-1}\left[\overline{t_n^2} + \overline{t_0^2} + (n\tau)^2 - 2n\overline{t_n}\tau + 2n\tau\overline{t_n}\right] - \frac{2}{M-1}\sum_{i=1}^{M}\left[t_{n,i}\cdot t_{0,i}\right] \tag{11}$$

Collecting like terms and substituting in Equation 4, we get

$$\sigma_n^2 = \frac{M}{M-1}\left[\overline{t_n{}^2} + \overline{t_0{}^2} - (n\tau)^2\right] - \frac{2}{M-1}\sum_{i=1}^{M}\left[t_{n,i}\cdot t_{0,i}\right] \tag{12}$$

Further substituting, using Equation 7, we get

$$\sigma_n^2 = \frac{2M\langle t \rangle}{M-1} - \frac{2}{M-1}\sum_{i=1}^{M}\left[t_{n,i}t_{0,i}\right] \tag{13}$$

The first term is a value independent of $n$, depending only on the size of the measurement ensemble and the expected value of jitter. To the extent that it does not vary over time (i.e., that the underlying phenomena are "stationary"), this term may be considered "constant." Thus, for stationary ergodic noise, the second term is proportional to the autocorrelation function, given by Equation 1.

Thus, we have shown that the variance as a function of time span (for jitter accumulation) as described by Equation 9, is equivalent to the autocorrelation function of the jitter.

## JITTER SPECTRA

The power spectral density (PSD)[4] is found by a Fourier transform of the autocorrelation function, which is known as the Blackman-Tukey method. The measured autocorrelation function of jitter is given by Equation 13. The PSD is found by performing a Fourier transform of $\sigma(n)^2$. To prevent a singularity at the origin, the mean of $\sigma_n^2$ is removed.

The jitter PSD has units of $t^2/\Delta f$, ($s^2$/Hz), where $t$ is the jitter, typically expressed in picoseconds, and $\Delta f$ is the resolution bandwidth in Hz. The domain of the PSD is the modulation frequency of the jitter. The frequency range of the jitter PSD depends on the maximum number of intervals used for the measurement of $\sigma_n$. The Nyquist frequency of the PSD measurement is given by

$$f_{Nyquist} = \frac{1}{2N\tau} \tag{14}$$

where $N$ is the maximum value for $n$ during data acquisition (or accumulation). For N=1, we get a Nyquist frequency equal to half the clock speed, $f_{Nyquist} = 0.5f_0$, which would not, in general, be very useful. The resolution bandwidth of the PSD for this kind of measurement is dependent on $N$ (and is equal to $1/(\tau N)$). And so $N$ must, practically speaking, be chosen to be large[5] in order to reveal periodic components that are small compared to the bit-rate.

The jitter spectrum is found by rescaling the PSD to compensate for the resolution bandwidth. This spectrum has units of jitter (i.e., time), typically expressed in picoseconds. For a clock signal, the jitter spectrum is similar to a phase noise spectrum[6]. Typical phase noise spectra do not have significant components above the Nyquist frequency ($f_0/2$).

For data signals, the methodology for finding a jitter spectrum is identical[7]. The variance of jitter accumulated over increasing unit intervals (UIs) is measured to give us the autocorrelation function, as described by Equation 13.

For nearly all standard compliance data patterns, edge-to-edge relationships for any integer span of UIs can easily be found[8], thus providing a uniformly sampled $\sigma_n$, as required for a discrete Fourier transform. In the rare[9] case where specific integer spans of UIs cannot be found in the data pattern, interpolation can be used to estimate the "missing" $\sigma_n$ values.

[4] Blackman, R. B., and Tukey, J.W., *The Measurement of Power Spectra,* Dover, 1958.

[5] As a practical matter, to detect periodic jitter in current-day systems (2004), it is common to measure to N > 2000 and sometimes as high as 20,000 UI.

[6] See application notes on phase noise posted by Aeroflex Incorporated on their website: www.aeroflex.com

[7] This document will not attempt to prove the validity of this assertion. However, using a Monte-Carlo model for jitter generation, Dr. Miller has confirmed the measurements to be correct, despite the presence of ISI and DCD and other forms of jitter.

8. Or interpolated

9. Some test patterns (the author believes inadvertently) are constructed so that some intervals of edge-to-edge times never occur.

Therefore, this measurement methodology can determine the jitter modulation spectra in the presence of a data signal, using the Blackman-Tukey method.

## Summary

A spectrum of jitter modulation can be found from clock or data signals by measuring the variance of jitter accumulated over increasing time spans. We have shown that this variance measurement is equivalent to the autocorrelation function of the jitter. The definitive "textbook" method for spectral analysis is the Blackman-Tukey method, which is to find the power spectral density using a Fourier transform of the autocorrelation function.



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